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On the energy of the de Sitter-Schwarzschild black hole

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ABSTRACT

Using Einstein's and Weinberg's energy complex, we evaluate the energy distribution of the vaccum nonsingularity black hole solution. The energy distribution is positive everywhere and be equal to zero at origin.

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There are two reasons for evaluting the energy of a system in general relativity. First, the conserved qualitities, like total energy etc., play important roles in solving the equation of motion. Second, the energy distribution must be positive if attractive force only. In particular positions, $r = 2M$ and $r = 0$, the Schwarzschild solution will be degenerate. The value $r = 2M$ is a removable coordinate singularity, however, the singularity at the origin is indeed irremoved. All of the physical laws are not holden at the singularity, hence we can't predict the evolution of singularity. The energy distribution at the singularity can't be prediction, therefore, the gravity theory without singularity is inevitable. Many theories, which are like Brans-Dicke theory, Einstein-Cartan theory etc., try to avoid the existence of the singularity, but they can not do it. With considering the action in which pure gravity term adds to cosmological constant term, the first vacuum nonsingularity solution is written down by de Sitter [1] in 1917. The de Sitter geometry is generated by a vacuum with nonzero energy density $\varepsilon = \frac{\Lambda}{8\pi}$, described by the stress-energy tensor

$$T_{\alpha\beta} = \varepsilon g_{\alpha\beta}, \tag{1}$$

with the equation of state

$$p = -\varepsilon. \tag{2}$$

Recent, Dymnikova [1] shows that the spherically symmetric vacuum can generate a black hole solution which is regular at $r = 0$ and everywhere else.

In the spherically symmetric static case, a line element has the form

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2d\theta^2 - r^2 sin\theta d\phi^2, \tag{3}$$

with the boundary conditions are the Schwarzschild behavior at $r \to \infty$,

$$ds^2 = (1 - \frac{r_g}{r})dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2d\theta^2 - r^2 sin\theta d\phi^2, \tag{4}$$

$$T_{\mu\nu} = 0, \tag{5}$$

where $r_g = 2M$, and the de Sitter behavior at $r \to 0$

$$ds^2 = (1 - \frac{r^2}{r_0^2})dt^2 - \frac{dr^2}{1 - \frac{r^2}{r_0^2}} - r^2d\theta^2 - r^2 sin\theta d\phi^2, \tag{6}$$

$$T_{\mu\nu} = \varepsilon_0 g_{\mu\nu}. \tag{7}$$

The limiting density is fixed and connected with the de Sitter horizon parameter r_0 by the de Sitter relation

$$r_0^2 = \frac{3}{8\pi\varepsilon_0}. \tag{8}$$

With the assumed form of the energy-momentum tensor

$$T_0^0 = \varepsilon_0 exp(-\frac{r^3}{r_0^2 r_g}), \tag{9}$$

they obtain the following metric

$$ds^2 = (1 - \frac{R_g(r)}{r})dt^2 - \frac{1}{1 - \frac{R_g(r)}{r}}dr^2 - r^2d\theta^2 - r^2 sin\theta d\phi^2. \tag{10}$$

In this solution, the defination of the parameters are

$$R_g(r) = r_g(1 - exp(\frac{r^3}{r_*^3})) \tag{11}$$

and

$$r_*^3 = r_0^2 r_g. \tag{12}$$

The structure of this solution, the so-called de Sitter-Schwarzschile solution, is like a Schwarzschild solution whose singularity is replaced by the de Sitter core. In this literary, we study the energy distribution of the de Sitter-Schwarzschild solution by the energy-momentum pseudotensor.

We consider the energy componens according to the energy-momentum pseudotensors of Einstein [2],

$$E(r) = \frac{1}{16\pi} \int \frac{\partial H_0^{0l}}{\partial x^l} d^3x, \tag{13}$$

where

$$H_0^{0l} = \frac{g_{00}}{\sqrt{-g}} \frac{\partial}{\partial x^m} \left[(-g) g^{00} g^{lm} \right], \tag{14}$$

and the energy-momentum pseudotensors of Weinberg [3],

$$E(r) = -\frac{1}{8\pi} \int \frac{\partial Q_0^{0l}}{\partial x^l} d^3x, \tag{15}$$

where

$$Q_0^{0l} = \frac{1}{2}(h_{lm,m} - h_{mm,l}), \tag{16}$$

$$h_{lm} = g_{lm} - \eta_{lm}. \tag{17}$$

The Latin index takes valuse from 1 to 3. We carry out the energy components according to energy-momentum pseudotensors of Einstein and Wein-

berg calculation in the quasi-Cartesian coordinate (t,x,y,z). The line element (10) converted into quasi-Cartesian coordinate is

$$ds^2 = (1-\frac{R_g(r)}{r})dt^2-(dx^2+dy^2+dz^2)-\left(\frac{R_g(r)}{r^2(r-R_g(r))}\right)(xdx+ydy+zdz)^2. \tag{18}$$

Thus, we obtain the required nonvanishing components of Einstein's energy-momentum pseudotensor H_0^{0l} in Eq. (14)

$$H_0^{01} = \frac{2x}{r^3}R_g(r), \tag{19}$$

$$H_0^{02} = \frac{2y}{r^3}R_g(r), \tag{20}$$

$$H_0^{03} = \frac{2z}{r^3}R_g(r). \tag{21}$$

Weinberg's energy-momentum pseudotensor Q_0^{0l} in Eqs. (16)-(17)

$$Q_0^{01} = -\frac{x}{r^3}R_g(r), \tag{22}$$

$$Q_0^{02} = -\frac{y}{r^3}R_g(r), \tag{23}$$

$$Q_0^{03} = -\frac{z}{r^3}R_g(r). \tag{24}$$

After plugging the nonvanishing components of energy-momentum pseudotensors into the formulation of energy complex, and applying the Gauss theorem, we evaluate the integral over the surface of a sphere with radius r.

Finally, we obtain the energy complex defined by Einstein and Weinberg within a sphere with radius r are the same, and the result is

$$E(r) = \frac{r_g}{2}\left(1-exp(-\frac{r^3}{r_*^3})\right), \tag{25}$$

and this result is the same as the standard formula for the mass [4]

$$\begin{aligned} m(r) &= 4\pi \int_0^r T_0^0 r^2 dr && (26) \\ &= \frac{r_g}{2}\left(1 - exp(-\frac{r^3}{r_*^3})\right). && (27) \end{aligned}$$

We plot the energy distributions of de Sitter-Schwarzschild black hole, see Figure . The behavior of energy distribution is like the Schwarzschild solution

$$E(r) = M \tag{28}$$

at $r \rightarrow \infty$, and the de Sitter solution (we show the energy distribution of de Sitter solition in the appendix of this literary.)

$$E(r) = \frac{r^3}{2r_0^2} \tag{29}$$

at $r \rightarrow 0$. By Einstein's and Weinberg's defination, we find that the energy distribution is positive everywhere, even in the region $r < r_H$, and be equal to zero at origin. This solution shows only attractive force everywhere including the origin in pure gravity theory.

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References

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[3] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of General Theory of Relativity*(John Wiley and Sons, Inc., NY, 1972).

[4] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, 1975).

Appendix

The de Sitter solution is the following metric

$$ds^2 = (1 - \frac{r^2}{r_0^2})dt^2 - \frac{dr^2}{1 - \frac{r^2}{r_0^2}} - r^2 d\theta^2 - r^2 sin\theta d\phi^2 \tag{30}$$

in spherical coordinate, and

$$ds^2 = (1 - \frac{r^2}{r_0^2})dt^2 - (dx^2 + dy^2 + dz^2) - (\frac{1}{r_0^2 - r^2})(xdx + ydy + zdz)^2 \tag{31}$$

in quasi-Cartesian coordinate, where $r_0^2 = \frac{3}{\Lambda}$ and Λ is the cosmological constant. According Eq.(14), we can obtain the nonvanishing components of Einstein's energy-momentum pseudotensor H_0^{0l}

$$H_0^{01} = \frac{2x}{r_0^2}, \tag{32}$$

$$H_0^{02} = \frac{2y}{r_0^2}, \tag{33}$$

$$H_0^{03} = \frac{2z}{r_0^2}. \tag{34}$$

Plugging those nonvanishing components into Eq.(13), and applying the Gauss theorem, we evaluate the integral over the surface of a sphere within radius r. The energy complex defined by Einstein within a sphere with radius r is

$$E(r) = \frac{2r^3}{r_0^2}. \tag{35}$$

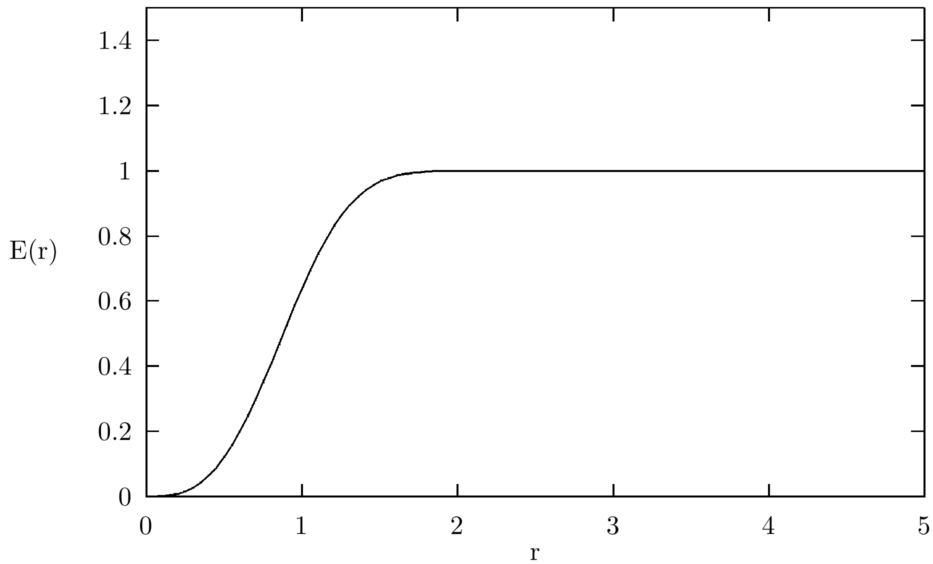

Figure 1: Thr energy distribution of de Sitter-Schwarzschild solution with $r_g = 2$ and $r_0 = \frac{1}{2}$.